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# Concepts of Lower Semicontinuity and Continuous Selections for Convex Valued Multifunctions*

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We discuss several concepts of continuity, weaker than lower semicontinuity, but still implying the existence of a continuous selection for a closed convex valued multifunction from a paracompact Hausdorff topological space into a Banach space. In this way, an extension of Michael's celebrated selection theorem is given. The behavior of $\varepsilon$-envelop approximations, as well as the localization of continuous selections, is also discussed. 

## 1. INTRODUCTION

Let $(Y, \|\cdot\|)$ be a normed linear space. Denote by $B$ the closed unit ball and by $B^o$ the open unit ball in $Y$. Let $\mathscr{P}(Y)$ denote the set of all subsets of $Y$ and let $\mathscr{N}(Y)=\mathscr{P}(Y)\setminus\{\varnothing\}$. For $y \in Y$, $C, D \in \mathscr{P}(Y)$ we write

$$y+C=\{y+v \mid v \in C\}, \qquad C+D=\{v+y \mid v \in C,\ y \in D\},$$
$$rC=\{rv \mid v \in C\} \qquad \text{for} \quad r \in \mathbb{R}.$$

The *distance* from $y$ to $C$ is defined by $\mathrm{d}(y, C)=\inf\{\|y-v\| \mid v \in C\}$, the *Hausdorff excess* of $D$ over $C$ by $\mathrm{e}(D, C)=\sup\{\mathrm{d}(y, C) \mid y \in D\}$, and the *Hausdorff distance* between the sets $C$ and $D$ by $\mathrm{D}(C, D)=\max\{\mathrm{e}(C, D), \mathrm{e}(D, C)\}$ (see [4]). It can be shown that

$$\mathrm{D}(C, D)=\sup_{y \in Y} |\mathrm{d}(y, C)-\mathrm{d}(y, D)|.$$

Moreover if $C_1 \subseteq C \subseteq C_2$ and $D_1 \subseteq D \subseteq D_2$ then

$$\mathrm{d}(y, C)-\mathrm{d}(y, D) \leqslant \mathrm{d}(y, C_1)-\mathrm{d}(y, D_2)$$

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and

$$\mathrm{D}(C, D) \leqslant \max\{\mathrm{D}(C_1, D_2), \mathrm{D}(C_2, D_1)\}.$$

Let $X$ be a topological space. By a set valued mapping $F$ of $X$ into $Y$ we mean a mapping $F: X \to \mathscr{P}(Y)$. By $\bar{F}$ we denote the mapping defined by $\bar{F}(x) = \overline{F(x)}$. Another set valued mapping $G: X \to \mathscr{P}(Y)$ is said to be a submapping of $F$, and we write $G \subseteq F$, if $G(x) \subseteq F(x)$ for every $x \in X$. By $F \cap G$ we denote the set valued mapping defined by $F \cap G(x) = F(x) \cap G(x)$. A function $f: X \to Y$ such that $f(x) \in F(x)$ ($f(x) \in F(x) + \varepsilon B^\circ$) for every $x \in X$ is called a selection ($\varepsilon$-approximate selection) of $F$. The set of all continuous selections for $F$ (respectively: all continuous $\varepsilon$-approximate selections) we denote by $C_F$ (respectively: $C_F^\varepsilon$). We use the same symbol for a set $K$ of functions from $X$ into $Y$ and the set valued mapping from $X$ into $Y$ defined by $K(x) = \{f(x) \mid f \in K\}$.

We introduce the definitions of certain set valued mappings derived from $F$ (see also [1–3]). For $x \in X$ let $\mathscr{U}(x)$ denote the family of all neighborhoods of $x$. For a set $D \in \mathscr{P}(Y)$ we define $F_D: X \to \mathscr{P}(Y)$ by

$$F_D(x) = \bigcup_{U \in \mathscr{U}(x)} \bigcap_{x' \in U} (F(x') - D)$$

and we write $F_\varepsilon$ instead of $F_{\varepsilon B^\circ}$ for $\varepsilon > 0$. We define $F_0: X \to \mathscr{P}(Y)$ by setting

$$F_0(x) = \bigcap_{\varepsilon > 0} F_\varepsilon(x).$$

$F_0(x)$ is a Kuratowski limes inferior of a net $(F(x) \mid x' \to x)$ and can be characterized as

$$F_0(x) = \{y \in \overline{F(x)} \mid \mathrm{d}(y, F(x')) \to 0 \text{ as } x' \to x\}.$$

It is easy to verify that $\bar{F}_0 = F_0 \subseteq \bar{F}$. Note that $F_{\{0\}} \subseteq F_0 \cap F$, but in general the equality does not hold.

Recall that $F: X \to \mathscr{N}(Y)$ is *lower semicontinuous* (l.s.c.) if $F^-(W) = \{x \mid F(x) \cap W \neq \varnothing\}$ is an open subset of $X$ for every open subset $W$ of $Y$ [13]; equivalently: if $F_0 = \bar{F}$ [2, Proposition 1.1; 1, Lemma 1.1], (see also Proposition 2.10 below). Following Deutsch and Kenderov [8] (see also [5]), we call $F$ *almost lower semicontinuous* (*almost l.s.c.*) if for every $x \in X$, $\varepsilon > 0$, it holds that $F_\varepsilon(x) \neq \varnothing$.

Now assume that $F: X \to \mathscr{N}(Y)$ has closed convex values, $X$ is a paracompact Hausdorff topological space, and $Y$ is a Banach space. The celebrated Michael theorem [13] asserts that if $F$ is lower semicontinuous then $F$ admits a continuous selection throughout each point of its graph, i.e., $C_F(x) = F(x)$ for all $x \in X$. Brown in [2] noted that $C_F = C_{F_0}$, thus if

$C_F \neq \emptyset$ then $F_0(x) \neq \emptyset$ for all $x \in X$. Deutsch and Kenderov [8] proved that $C_F^\varepsilon \neq \emptyset$ for every $\varepsilon > 0$ if and only if $F$ is almost l.s.c.

Moreover if $F$ has convex values then $F_\varepsilon$ and $F_0$ also have convex values (possible empty). As we remarked $F_0 = \bar{F}_0$. Thus, if $F_0(x) \neq \emptyset$ for all $x \in X$ and $(F_0)_0 = F_0$, i.e., $F_0$ is l.s.c., then $C_F \neq \emptyset$, by virtue of Michael's theorem. In a recent paper [3] Brown defined a transfinite sequence $(F^{(\alpha)} \mid \alpha$ an ordinal) by letting

$$F^{(0)} = F, \qquad F^{(\alpha+1)} = (F^{(\alpha)})_0, \qquad F^{(\beta)}(x) = \bigcap_{\alpha < \beta} F^{(\alpha)}(x)$$

whenever $\beta$ is a limit ordinal. He observed that there must occur the first ordinal $\gamma$ such that $F^{(\gamma+1)} = F^{(\gamma)}$. Since $C_F = C_{F^{(\alpha)}}$ and $F^{(\alpha)}$ has closed convex values for every ordinal $\alpha$, Brown concluded that $C_F \neq \emptyset$ if and only if $F^{(\gamma)}(x) \neq \emptyset$ for all $x \in X$. Brown proved also that in the case $Y = \mathbb{R}^n$ it holds that $C_F \neq \emptyset$ if and only if $F^{(n)}(x) \neq \emptyset$ for all $x \in X$. But it may happen that $F^{(n)}(x) \neq \emptyset$ for all $x \in X$ and $F^{(n)} \neq F^{(n-1)}$ [3, Theorems 4.3, 1.3].

We are concerned with the question: When for $F: X \to \mathcal{P}(Y)$ can one assert that $F_0(x) \neq \emptyset$ for all $x \in X$ and $(F_0)_0 = F_0$? The problem may be transferred naturally to every $F^{(\alpha)}$.

In Sections 2 and 3 we discuss several criteria for $F$ to yield a positive answer. Our consideration covers the "separation type" concepts of lower semicontinuity which extend the ideas from [6, 15], as well as the convergence approach involving the mappings $F_\varepsilon$ due to Beer [1] and the conditions introduced in [7, 9]. We present also the results on approximation of the submapping $F_0$. In Section 4 we use these results in approximating the set of continuous selections for $F$. The relevance of the mappings $F_\varepsilon$ to the study of continuous selections for $F$ is due to the simply verifiable relations

$$C_{F_\varepsilon} \subseteq C_F^\varepsilon \subseteq C_{F_\delta} \qquad \text{for} \quad 0 < \varepsilon < \delta \text{ and } \bigcap_{\varepsilon > 0} C_F^\varepsilon = C_F$$

(see Proposition 4.1 below). Section 5 is devoted to the study of localization of continuous selections. We discuss in this section the heredity of the properties introduced for intersections of a set valued mapping with ball valued mappings. Finally, Section 6 contains some comment on the selection extension property.

If it is not assumed otherwise, throughout the paper $X$ is a topological space and $Y$ is a normed linear space.

## 2. The Concepts of Lower Semicontinuity

In this section we introduce the lower semicontinuity conditions for $F$. We elucidate interrelations between them and their relation to the lower semicontinuity of $F$.

First we list some elementary properties of the derived mappings $F_D$, $F_\varepsilon$, for $\varepsilon \geqslant 0$, defined in the Introduction. Recall that $F: X \to \mathscr{P}(Y)$ is said to have *open lower sections* if $F^-(W)$ is an open subset of $X$ for every subset $W$ of $Y$ (see, e.g., [18]). Clearly $F: X \to \mathscr{N}(Y)$ has open lower sections if for every $x \in X$, $F$ has open lower sections at $x$, i.e., $F(x) \cap W \neq \varnothing$ implies $0 \in F_W(x)$, for every subset $W$ of $Y$.

2.1. Proposition. *Let $F, G: X \to \mathscr{P}(Y)$ be set valued mappings and let $x \in X$, $z \in Y$, $D, E \subseteq Y$. Then:*

(1) $z \in F_D(x)$ *if and only if* $0 \in F_{z+D}(x)$,

(2) $F_{\{z\}}(x) \cap D \neq \varnothing$ *implies* $z \in F_D(x)$,

(3) $F_D(x) \subseteq F(x) - D$,

(4) $F_D$ *has open lower sections,*

(5) $(F_D)_E(x) = F_D(x) - E \subseteq F_{D+E}(x)$ *for all* $x \in X$,

(6) $D \subseteq E$ *implies* $F_D \subseteq F_E$,

(7) *if* $D + A = E$ *for some* $A \subseteq Y$ *then* $(F_D)_E \subseteq (F_E)_D$,

(8) $F \subseteq G$ *implies* $F_D \subseteq G_D$.

*Proof.* Parts (1), (2), (3), (6), and (8) are immediate consequences of the definitions.

(4) First observe that $F_D^-(\{z\}) = \{x \in X \mid z \in F_D(x)\}$ is an open subset of $X$ for every $z \in Y$. Indeed, if $x_0 \in F_D^-(\{z\})$ then there exists $U \in \mathscr{U}(x_0)$ such that $F(x') \cap (z + D) \neq \varnothing$ for all $x' \in U$. Then for every $x \in U$ we have $U \in \mathscr{U}(x)$ and $F(x') \cap (z + D) \neq \varnothing$ for all $x' \in U$, hence $U \subseteq F_D^-(\{z\})$. Consequently, $F_D^-(W) = \bigcup_{z \in W} F_D^-(\{z\})$ is an open subset of $X$ for arbitrary $W \subseteq Y$.

(5) From (3) applied to $F_D$ it follows that $(F_D)_E(x) \subseteq F_D(x) - E$. For the reverse inclusion note that if $z \in F_D(x) - E$, i.e., $F_D(x) \cap (z + E) \neq \varnothing$, then since $F_D$ has open lower sections there exists $U \in \mathscr{U}(x)$ such that $F_D(x') \cap (z + E) \neq \varnothing$ for all $x' \in U$, i.e., $z \in (F_D)_E(x)$. To complete (5) observe that $z \in F_D(x) - E$ if and only if $w \in F_D(x)$ for some $w \in z + E$. Clearly $w + D \subseteq z + E + D$, and thus we conclude that $z \in F_{D+E}(x)$.

(7) If $E = D + A$ then by (5)

$$(F_D)_E(x) = F_D(x) - A - D = (F_D)_A(x) - D \subseteq F_{D+A}(x) - D = (F_E)_D(x),$$

hence (7) holds. Q.E.D.

2.2. PROPOSITION. *Let $F: X \to \mathscr{P}(Y)$ be a set valued mapping and $0 < \delta \leqslant \varepsilon$. Then*

$$(F_\delta)_\varepsilon(x) = F_\delta(x) - \varepsilon B^\circ \subseteq F_\varepsilon(x) - \delta B^\circ = (F_\varepsilon)_\delta(x) \subseteq F_{\varepsilon+\delta}(x)$$

*for all $x \in X$. Moreover $\bar{F}_0 = F_0$ and*

$$(F_0)_\varepsilon(x) \subseteq F_0(x) - \varepsilon B^\circ \subseteq \overline{F_\varepsilon(x)} = (F_\varepsilon)_0(x)$$

*for all $x \in X$.*

*Proof.* The first statement is a direct consequence of Proposition 2.1(5), (7). That $(F_0)_\varepsilon(x) \subseteq F_0(x) - \varepsilon B^0$ follows by Proposition 2.1(3) applied to $F_0$. Since

$$F_0(x) - \varepsilon B^\circ \subseteq F_\delta(x) - \varepsilon B^\circ \subseteq F_\varepsilon(x) - \delta B^\circ \subseteq F_{\varepsilon+\delta}(x)$$

for all $0 < \delta \leqslant \varepsilon$, it follows that $\overline{F_0(x)} = F_0(x)$ and that $F_0(x) - \varepsilon B^\circ \subseteq \overline{F_\varepsilon(x)}$. But by Proposition 2.1(5), (6) we have

$$F_\varepsilon(x) = (F_\varepsilon)_{\{0\}}(x) \subseteq \bigcap_{\delta > 0} (F_\varepsilon)_\delta(x) = (F_\varepsilon)_0(x) \subseteq \overline{F_\varepsilon(x)}.$$

Thus $\overline{F_\varepsilon(x)} = \overline{(F_\varepsilon)_0(x)} = (F_\varepsilon)_0(x)$. Q.E.D.

2.3. PROPOSITION. *If $F: X \to \mathscr{P}(Y)$ has convex values and $D$ is a convex subset of $Y$ then $F_D$ and $F_0$ have convex values.*

We say that $F: X \to \mathscr{N}(Y)$ is *weak lower semicontinuous* (*weak l.s.c.*) at $x$ if

*for every $\varepsilon > 0$ and $U \in \mathscr{U}(x)$ there exists $x' \in U$ such that $F(x') \subseteq F_\varepsilon(x)$.*

This concept, following the concept of weak Hausdorff lower semicontinuity due to De Blasi and Myjak [6], was introduced by the authors [15], and under the more fortunate name *quasi-lower semicontinuity* by Gutev [11].

We say that $F: X \to \mathscr{N}(Y)$ is *convex lower semicontinuous* (*convex l.s.c.*) at $x$ if

*$0 \in F_D(x)$ implies $F_\varepsilon(x) \cap D \neq \varnothing$, for all $\varepsilon > 0$ and all closed convex subsets $D$ of $Y$.*

Let $K \geqslant 1$. We say that $F: X \to \mathscr{N}(Y)$ is *K-ball Lipschitz lower semicontinuous* (*K-ball-Lipschitz l.s.c.*) at $x \in X$, if

*$F_\varepsilon(x) \neq \varnothing$ whenever $\varepsilon > 0$, and*

*$0 \in F_{y+rB}(x)$ implies $F_\varepsilon(x) \cap (y + KrB) \neq \varnothing$, for all $y \in Y$ and all $r \geqslant 0$, $\varepsilon > 0$.*

We say that $F: X \to \mathcal{N}(Y)$ is *weak l.s.c.*, (*convex l.s.c.*, *K-ball-Lipschitz l.s.c.*) if $F$ is weak l.s.c. (convex l.s.c., $K$-ball-Lipschitz l.s.c.) at every $x \in X$. We say simply that $F$ is *ball-Lipschitz l.s.c.* if $F$ is $K$-ball-Lipschitz l.s.c. for some $K \geqslant 1$.

We say that $F: X \to \mathcal{N}(Y)$ is *ball-uniformly lower semicontinuous* (*ball-uniformly l.s.c.*), if

*$F_\varepsilon(x) \neq \varnothing$ whenever $\varepsilon > 0$, and*

*for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $F_\delta(x) \subseteq F_\mu(x) - \varepsilon B^\circ$ for all $\mu > 0$, and for all $x \in X$.*

2.4. THEOREM. *Let $F: X \to \mathcal{N}(Y)$ be a set valued mapping. Consider the following statements*:

(1) *$F$ has open lower sections,*

(2) *$F$ is l.s.c.,*

(3) *$F$ is weak l.s.c.,*

(4) *$F$ is convex l.s.c.,*

(5) *$F$ is* 1-*ball-Lipschitz l.s.c.,*

(6) *$F$ is ball-Lipschitz l.s.c.,*

(7) *$F$ is ball-uniformly l.s.c.,*

(8) *$F$ is almost l.s.c.*

*Then* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8)$.

*Proof.* $(1) \Rightarrow (2)$, $(5) \Rightarrow (6)$, $(7) \Rightarrow (8)$ are obvious.

$(2) \Rightarrow (3)$. Clearly $F$ is l.s.c. if and only if $F(x) \cap W \neq \varnothing$ implies $0 \in F_W(x)$, for every $x \in X$ and every open set $W \subseteq Y$. Therefore if $(z \in \varepsilon B^\circ) \cap F(x) \neq \varnothing$, then $0 \in F_{z+\varepsilon B^\circ}(x)$, i.e., $z \in F_\varepsilon(x)$ (Proposition 2.1(1)). Thus $F(x) - \varepsilon B^\circ \subseteq F_\varepsilon(x)$.

$(3) \Rightarrow (4)$. For arbitrary subset $D$ of $Y$, if $0 \in F_D(x)$ then there exists $U \in \mathscr{U}(x)$ such that $F(x') \cap D \neq \varnothing$ for all $x' \in U$. By (3), for every $\varepsilon > 0$ there exists $x_\varepsilon \in U$ such that $\varnothing \neq F(x_\varepsilon) \cap D \subseteq F_\varepsilon(x) \cap D$.

$(4) \Rightarrow (5)$. We need only observe that $F_\varepsilon(x) \neq \varnothing$ since $0 \in F_Y(x)$.

$(6) \Rightarrow (7)$. Suppose that $y \in F_\varepsilon(x)$. Then $0 \in F_{y+\varepsilon B}(x)$ and therefore, by the assumption, $F_{2^{-1}\mu}(x) \cap (y + K\varepsilon B) \neq \varnothing$, or equivalently $y \in F_{2^{-1}\mu}(x) - K\varepsilon B$, for every $\mu > 0$. Thus by Proposition 2.2, we get

$$F_\varepsilon(x) \subseteq F_{2^{-1}\mu}(x) - K\varepsilon B \subseteq F_{2^{-1}\mu}(x) - 2^{-1}\mu B^\circ - K\varepsilon B^\circ \subseteq F_\mu(x) - K\varepsilon B,$$

for every $\mu > 0$. Q.E.D.

Clearly $F: X \to \mathcal{N}(Y)$ may be l.s.c. but may have not open lower sections, e.g., $F(x) = \{x\}$ for $x \in \mathbb{R}$. Two examples of set valued mappings $F: \mathbb{R} \to \mathcal{N}(\mathbb{R})$ with compact convex values, one of which is weak l.s.c. but not l.s.c., and the other is convex l.s.c. but not weak l.s.c. at every $x \in \mathbb{R}$, are given in [15, Examples 2, 4]. Note that in the case of 1-dimensional space $Y$ the following coincidence holds.

2.5. PROPOSITION. *Assume that the set valued mapping $F: X \to \mathcal{N}(\mathbb{R})$ has convex values. Then F is almost l.s.c. if and only if F is convex l.s.c.*

*Proof.* Assume that $F$ is almost l.s.c. and let $D$ be a closed convex subset of $\mathbb{R}$, $\varepsilon > 0$, $x \in X$. Clearly $F_\varepsilon(x)$ is a nonempty interval, possibly infinite. If $F_\varepsilon(x) \cap D = \emptyset$ then for every $U \in \mathcal{U}(x)$ there exists $x' \in U$ such that $\inf D \geqslant \sup(F(x') - \varepsilon B^o)$ or $\sup D \leqslant \inf(F_\varepsilon(x') - \varepsilon B^o)$. Hence $F(x') \cap D = \emptyset$. Consequently $0 \notin F(x)$. Q.E.D.

Thus the lower semicontinuity conditions (4)–(8) in Theorem 2.4 are equivalent for a convex valued mapping $F: X \to \mathcal{N}(\mathbb{R})$. The following simple examples show that these concepts are distinct even for a convex valued mapping $F: \mathbb{R} \to \mathcal{N}(\mathbb{R}^2)$.

2.6. EXAMPLE. Let $F(0) = \{(0, t) \mid t \geqslant 0\}$, $F(x) = \{(|x|^{-1} xs, |x| s) \mid s \geqslant 0\}$ for $x \neq 0$. Note that $F_\varepsilon(0) = \varepsilon B^\circ \cup \{(0, \varepsilon)\}$ for every $\varepsilon \geqslant 0$. For any closed ball $D \subset Y$ with the property $F_\varepsilon(0) \cap D = \emptyset$ there exists a closed halfspace $H$ containing $D$ and such that the set $F_\varepsilon(0) \cap H$ is empty or consists of a single point $(0, \varepsilon)$. If it happens that $0 \in F_D(0)$, then $0 \in F_H(0)$, which together with $F_\varepsilon(0) \cap H \subseteq \{(0, \varepsilon)\}$ implies that $H = H_\alpha$ for some $\alpha \geqslant 0$, where $H_\alpha = \{(s, t) \mid t \geqslant \alpha\}$. However, for a bounded subset $D$ of such a halfspace it must hold that $0 \notin F_D(0)$. Thus, $F$ is 1-ball-Lipschitz l.s.c. at 0. Finally, $F$ is not convex l.s.c. at 0, since for $0 < \varepsilon < \alpha$ we have $0 \in F_{H_\alpha}(0)$ and $F_\varepsilon(0) \cap H_\alpha = \emptyset$.

2.7. EXAMPLE. Let $F(x) = \{(t, 0) \mid 0 \leqslant t \leqslant 1\}$ for $x$ irrational and $F(x) = \{(t, t \tan \alpha) \mid 0 \leqslant t \leqslant 1\}$ for $x$ rational, where $\alpha \in (0, \pi]$. Then $K = (\sin(\alpha/2))^{-1}$ is the smallest number such that $F$ is $K$-ball-Lipschitz l.s.c.

2.8. EXAMPLE. Let $F(x) = \{(t, 0) \mid 0 \leqslant t \leqslant 1\}$ for $x$ irrational and $F(x) = \{(t, s) \mid 0 \leqslant t \leqslant 1, t^2 \leqslant s \leqslant 1\}$ for $x$ rational. $F$ is not $K$-ball-Lipschitz l.s.c. but is ball-uniformly l.s.c.

It is worth noting that the weak lower semicontinuity is a property which is possessed by almost l.s.c. metric projections onto finite dimensional subspaces in the space of continuous functions. Actually for such set valued mappings the stronger continuity conditions were proved, via certain perturbation theorems, and then used in proving the existence of continuous selections (see [9, 17, 2] and references therein). We cite below only the relevant conclusion from Wu Li's paper [17].

2.9. PROPOSITION (17, Corollary 4.4, Lemma 3.5]. *Let $X$ be a Banach space of continuous functions from a locally compact Hausdorff space into a strictly convex Banach space which vanish at infinity. Let $F$ be the metric projection from $X$ onto a finite dimensional subspace of $X$. Then for any $x \in X$ and for any $\varepsilon > 0$ there exists $x' \in x + \varepsilon B^\circ$ such that $F(x') \subseteq F(x)$ and $F$ is l.s.c. at $x'$. If $F$ is almost l.s.c. then for any $x \in X$ and for any $\varepsilon > 0$ there exists $x' \in x + \varepsilon B^\circ$ such that $F(x') \subseteq F_0(x)$.*

From the second assertion of Proposition 2.9 it follows that such a metric projection $F$ is almost l.s.c. if and only if it is weak l.s.c. In general, it may happen that a set valued mapping has compact convex values in euclidean space $\mathbb{R}^2$, is almost l.s.c. and upper semicontinuous on the whole domain, as well as l.s.c. on a dense subset of the domain, but has no continuous selection (see [3, Theorem 4.2a; 19, Sect. 3; 1, Example 1]; see also [8]). We postpone the discussion on continuous selections of the maps satisfying our lower semicontinuity conditions to Section 4. In the propositions below we list more characterizations of these concepts.

We assume that $F: X \to \mathcal{N}(Y)$ and $x \in X$.

2.10. PROPOSITION. *The following statements are equivalent:*

(1) *$F$ has open lower sections at $x$,*

(2) *$F(x) \cap D \neq \emptyset$ implies $F_{\{0\}}(x) \cap D \neq \emptyset$, for every $D \subseteq Y$,*

(3) *$F(x) = F_{\{0\}}(x)$.*

2.11. PROPOSITION. *The following statements are equivalent:*

(1) *$F$ is l.s.c. at $x$,*

(2) *$F(x) \cap D \neq \emptyset$ implies $F_\varepsilon(x) \cap D \neq \emptyset$, for every $D \subseteq Y$, $\varepsilon > 0$,*

(3) *$F(x) - \varepsilon B^\circ = F_\varepsilon(x)$ for every $\varepsilon > 0$,*

(4) *$\overline{F(x)} = F_0(x)$.*

2.12. PROPOSITION. *The following statements are equivalent:*

(1) *$F$ is weak l.s.c. at $x$,*

(2) *$0 \in F_D(x)$ implies $F_\varepsilon(x) \cap D \neq \emptyset$, for all $\varepsilon > 0$ and all subsets $D$ of $Y$,*

(3) *$0 \in F_D(x)$ implies $F_\varepsilon(x) \cap D \neq \emptyset$, for all $\varepsilon > 0$ and all closed subsets $D$ of $Y$.*

2.13. PROPOSITION. *If F has convex values then the following statements are equivalent*:

(1) *F is convex l.s.c. at x*,

(2) $0 \in F_D(x)$ *implies* $F_\varepsilon(x) \cap D \neq \varnothing$, *for all* $\varepsilon > 0$ *and all closed halfspaces including the improper case* $D = Y$.

2.14. PROPOSITION. *Let* $K \geqslant 1$. *The following statements are equivalent*:

(1) *F is K-ball-Lipschitz l.s.c. at x*,

(2) $\varnothing \neq F_\varepsilon(x) \subseteq F_\mu(x) - K\varepsilon B^\circ$ *for all* $\varepsilon, \mu > 0$.

2.15. PROPOSITION. *The following statements are equivalent*:

(1) *F is* 1*-ball-Lipschitz l.s.c. at x*,

(2) $\varnothing \neq \overline{F_\varepsilon(x) - \delta B^\circ} = \overline{F_\delta(x) - \varepsilon B^\circ}$ *for all* $\delta, \varepsilon > 0$.

*Moreover, if F has convex values then these statements are equivalent to*:

(3) $\varnothing \neq F_\varepsilon(x) - \delta B^\circ = F_\delta(x) - \varepsilon B^\circ$ *for all* $\delta, \varepsilon > 0$.

## 3. THE CONVERGENCE OF THE NET $(F_\varepsilon \mid \varepsilon \searrow 0)$

We give a sufficient condition for $F_0$ to be l.s.c., in terms of—locally uniformly for arguments and for values—a ball lower semicontinuity type condition for $F$. Namely, we say that $F: X \to \mathcal{N}(Y)$ is *ball-locally-uniformly l.s.c.* if

*every* $x \in X$ *has a neighborhood U such that for every* $y \in Y$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with the property that for every* $x' \in U$ *there exists* $r \geqslant 0$ *such that*

$$\varnothing \neq (y + rB^\circ) \cap F_\delta(x') \subseteq F_\mu(x') - \varepsilon B^\circ$$

*for all* $\mu > 0$.

Clearly if $F$ is ball-uniformly l.s.c. then $F$ is ball-locally-uniformly l.s.c. Since $F_\gamma(x) \subseteq F_\beta(x)$ for all $x \in X$, $\gamma \leqslant \beta$, the ball-uniformly lower semicontinuity means that $\sup_x \mathrm{D}(F_\gamma(x), F_\beta(x)) \to 0$ as $\gamma, \beta \to 0$, i.e., the decreasing nets $(F_\varepsilon(x) \mid \varepsilon \searrow 0)$ are Cauchy nets in the generalized metric space $(\mathcal{N}(Y), \mathrm{D})$, uniformly for $x \in X$, while the ball-locally-uniformly lower semicontinuity means that these nets are locally Cauchy nets in $(\mathcal{N}(Y), \mathrm{D})$, locally uniformly for $x \in X$. The proof of our next theorem invokes the construction and involves an extension (probably known) of the classical result on the Cauchy sequences of closed sets in complete metric spaces due to Hahn [12]; see also [4, Theorem II-3].

3.1. THEOREM. *Let $Y$ be a Banach space. Assume that a set valued mapping $F: X \to \mathcal{N}(Y)$ is ball-locally-uniformly l.s.c. Then $F_0(x) \neq \varnothing$ for all $x \in X$ and $F_0$ is l.s.c. Moreover, for every $x \in X$, $y \in Y$, and $\varepsilon > 0$ there exist a neighborhood $V$ of $x$ and $\delta > 0$ such that*

$$d(y, F_\delta(x')) \leqslant d(y, F_0(x')) < d(y, F_\delta(x')) + \varepsilon$$

*for all $x' \in V$.*

*Proof.* Observe that from ball-locally-uniformly l.s.c. it follows immediately that $F_\varepsilon(x) \neq \varnothing$ whenever $\varepsilon > 0$, for all $x \in X$. First we will show that $F_0(x) \neq \varnothing$ for every $x \in X$. To this end, for each $x \in X$ and for every $\varepsilon > 0$ we will define inductively a sequence $(y_n)$ such that

(1) $(y_n + 2^{-n-1}\varepsilon B^\circ) \cap F_\mu(x) \neq \varnothing$ *for all* $\mu > 0$, $n = 1, 2, \ldots$,

(2) $y_n \in y_{n-1} + 2^{-n-1}\varepsilon B^\circ$ *for* $n = 2, 3, \ldots$ .

Fix an arbitrary $y \in Y$ and choose $\delta > 0$ and $r \geqslant 0$ such that

$$\varnothing \neq (y + rB^\circ) \cap F_\delta(x) \subseteq F_\mu(x) - 2^{-2}\varepsilon B^\circ$$

for all $\mu > 0$. Let $d = d(y, F_\delta(x)) + 2^{-1}\varepsilon$. Choose $y_1 \in (y + dB^\circ) \cap (y + rB^\circ) \cap F_\delta(x)$. Clearly

$$(y_1 + 2^{-2}\varepsilon B^\circ) \cap F_\mu(x) \neq \varnothing \qquad \text{for all} \quad \mu > 0.$$

Assume that for $n = 1, \ldots, k$ we have defined $y_n$ satisfying (1) and (2). For $y_k$ choose $\delta_k > 0$ and $r_k \geqslant 0$ such that

$$\varnothing \neq (y_k + r_k B^\circ) \cap F_{\delta_k}(x) \subseteq F_\mu(x) - 2^{-k-2}\varepsilon B^\circ$$

for all $\mu > 0$. Since $(y_k + 2^{-k-1}\varepsilon B^\circ) \cap F_{\delta_k}(x) \neq \varnothing$ we can choose $y_{k+1} \in (y_k + 2^{-k-1}\varepsilon B^\circ) \cap (y_k + r_k B^\circ) \cap F_{\delta_k}(x)$. Then

$$(y_{k+1} + 2^{-k-2}\varepsilon B^\circ) \cap F_\mu(x) \neq \varnothing$$

for all $\mu > 0$. Thus (1) and (2) are satisfied for $n = k + 1$. Note now that from (1) and (2) it follows that $(y_n)$ is a Cauchy sequence, hence convergent to some $y_0$, and $y_0 \in y_n + 2^{-n}\varepsilon B^\circ$ for every $n$. Since

$$y_n \in \bigcap_{\mu > 0} (F_\mu(x) - 2^{-n-1}\varepsilon B^\circ) \subseteq \bigcap_{\mu > 0} F_{\mu + 2^{-n-1}\varepsilon}(x),$$

it follows that

$$y_0 \in \bigcap_n \left( \left( \bigcap_{\mu > 0} F_{\mu + 2^{-n-1}\varepsilon}(x) \right) - 2^{-n}\varepsilon B^\circ \right)$$
$$\subseteq \bigcap_n \bigcap_{\mu > 0} F_{\mu + (2^{-n-1} + 2^{-n})\varepsilon}(x) = F_0(x).$$

Moreover,

$$\begin{aligned} d(y, F_\delta(x)) &\leqslant d(y, F_0(x)) \leqslant \|y - y_1\| + \|y_1 - y_0\| \\ &< d(y, F_\delta(x)) + 2^{-1}\varepsilon + 2^{-1}\varepsilon. \end{aligned}$$

Since $y \in Y$ was arbitrarily chosen, from the first inequality it follows in particular that the net $(F_\delta(x) \mid \delta \searrow 0)$ is $V^{-1}$-convergent to $F_0(x)$, where $V^{-1}$ stands for the lower Vietoris topology on $\mathcal{N}(Y)$ [10, Proposition 2.1]. The same inequalities are valid for all $x'$ from some neighborhood $U$ of $x$ and the sequences $(y'_n)$ defined in this way for $x'$ and $y$. Therefore for $x' \in U$ and $y' \in y + 2^{-1}\varepsilon B^o$ we have

$$\begin{aligned} d(y, F_\delta(x')) &\leqslant d(y, F_0(x')) \leqslant \|y - y'_1\| + \|y'_1 - y'_0\| \\ &< d(y, F_\delta(x')) + 2^{-1}\varepsilon + 2^{-1}\varepsilon. \end{aligned}$$

Thus, in particular, for arbitrary $\nu > \mu > 0$, $\beta \leqslant \delta$, and $x' \in U$ we have

$$F_0(x') \cap (y + \nu B^o) \neq \varnothing$$

whenever

$$F_\beta(x') \cap (y + (\nu - \mu) B^o) \neq \varnothing.$$

From this it follows immediately that the net $(F_\delta \mid \delta \searrow 0)$ is $\mathcal{SW}^{-1}$-quasi-locally uniformly convergent to $F_0$ at $x$ for the quasi-uniformity $\mathcal{SW}^{-1}$ on $\mathcal{N}(Y)$ associated with the norm uniformity on $Y$, in the sense of [10, Sect. 7]. Since all $F_\delta$ are l.s.c., it follows that $F_0$ is l.s.c., by virtue of [10, Theorem 7.3]. Q.E.D.

Theorem 3.1 extends the result due to Beer [1, Theorem 1], who assumed locally uniform convergence of the $F_{1/n}$ to $F_0$ in Hausdorff metric, and showed that $F_0$ is l.s.c. Note that by the same arguments as those in Theorem 3.1 we have the following.

3.2. COROLLARY. *If $Y$ is a Banach space and $F: X \to \mathcal{N}(Y)$ is ball-uniformly l.s.c., then $F_0(x) \neq \varnothing$ for all $x \in X$, $F_0$ is l.s.c., and $\sup_x D(F_\gamma(x), F_0(x)) \to 0$ as $\gamma \to 0$. If $F$ is $K$-ball-Lipschitz l.s.c., then $\sup_x D(F_\gamma(x), F_0(x)) \leqslant K\gamma$ for all $\gamma \geqslant 0$.*

Indeed, the first assertion is clear, and for the second one it suffices to recall that $K$-ball-Lipschitz lower semicontinuity of $F$ means that $\sup_x D(F_\gamma(x), F_\beta(x)) \leqslant K \max\{\gamma, \beta\}$ for $\gamma, \beta > 0$.

Now we take advantage of the convexity of the values of $F$ in approximating the submapping $F_0$ by $F_\varepsilon$. The following simple property is fundamental for our considerations.

3.3. PROPOSITION. *If a set valued mapping* $F: X \to \mathcal{N}(Y)$ *has convex values then for all* $x \in X$ *the maps* $\varepsilon \to F_\varepsilon(x)$ *have convex graphs.*

*Proof.* Let $x \in X$, $0 \leqslant \delta \leqslant \varepsilon$, $0 \leqslant s \leqslant 1$, $y_\delta \in F_\delta(x)$, $y_\varepsilon \in F_\varepsilon(x)$, and $y = (1-s)\, y_\delta + s y_\varepsilon$. There exists $U \in \mathcal{U}(x)$ such that for all $x' \in U$

$$y_\delta \in F(x') - \delta B^\circ \qquad \text{and} \qquad y_\varepsilon \in F(x') - \varepsilon B^\circ.$$

Thus

$$\begin{aligned} y &= (1-s)\, y_\delta + s y_\varepsilon \in (1-s)(F(x') - \delta B^\circ) + s(F(x') - \varepsilon B^\circ) \\ &= F(x') - ((1-s)\,\delta + s\varepsilon)\, B^\circ, \end{aligned}$$

for all $x' \in U$. Hence $y \in F_{(1-s)\delta + s\varepsilon}(x)$. Q.E.D.

Observe that by Proposition 3.3 the functions $\varepsilon \to \mathrm{d}(y, F_\varepsilon(x))$, $x \in X$, $y \in Y$, are convex. Clearly

$$\mathrm{D}(F_\varepsilon(x), F_\delta(x)) = \sup_y |\mathrm{d}(y, F_\varepsilon(x)) - \mathrm{d}(y, F_\delta(x))|.$$

For $0 \leqslant \delta \leqslant \varepsilon$ we have $F_\delta(x) - (\varepsilon - \delta)\, B^\circ \subseteq F_\varepsilon(x)$ and $\mathrm{d}(y, F_\delta(x)) = \mathrm{d}(y, F_\delta(x) - (\varepsilon - \delta)\, B^\circ) + (\varepsilon - \delta)$, whenever $y \notin F_\varepsilon(x)$. Thus for $\varepsilon \geqslant \delta \geqslant 0$ we have

$$\mathrm{d}(y, F_\delta(x)) - \mathrm{d}(y, F_\varepsilon(x)) \geqslant \varepsilon - \delta,$$

whenever $y \notin F_\varepsilon(x)$ and

$$\mathrm{D}(F_\varepsilon(x), F_\delta(x)) \geqslant \varepsilon - \delta$$

whenever $F_\varepsilon(x) \neq Y$. The next proposition summarizes the above observations. We denote by $\partial_\varepsilon^+ \mathrm{d}(y, F_\beta(x))$ the right-hand derivative of $\varepsilon \to \mathrm{d}(y, F_\varepsilon(x))$ at $\beta$.

3.4. PROPOSITION. *Assume that a set valued mapping* $F: X \to \mathcal{N}(Y)$ *has convex values, is almost l.s.c. at* $x \in X$ *and* $F(x) \neq Y$. *Then for every* $y \in Y$ *the function* $\varepsilon \to \mathrm{d}(y, F_\varepsilon(x))$ *is a finite convex nonincreasing function on* $(0, +\infty)$ *and* $-\infty < \partial_\varepsilon^+ \mathrm{d}(y, F_\varepsilon(x)) \leqslant -1$ *whenever* $y \notin \overline{F_\varepsilon(x)}$. *Moreover*

$$|\mathrm{d}(y, F_\varepsilon(x)) - \mathrm{d}(y, F_\delta(x))| \leqslant |\partial_\varepsilon^+ \mathrm{d}(y, F_\beta(x))|\, |\varepsilon - \delta|$$

*and*

$$|\varepsilon - \delta| \leqslant \mathrm{D}(F_\varepsilon(x), F_\delta(x)) \leqslant \lim_{\gamma \searrow \beta} \frac{\mathrm{D}(F_\gamma(x), F_\beta(x))}{\gamma - \beta}\, |\varepsilon - \delta|$$

*whenever* $0 \leqslant \beta \leqslant \delta,\ \varepsilon$.

By Proposition 3.4, Corollary 3.2, and Lipschitz properties of convex functions (see [16]), we have the following.

3.5. Corollary. *Assume that Y is a Banach space and* $F: X \to \mathcal{N}(Y)$ *assumes convex values. If F is ball-uniformly l.s.c. then the mappings* $\varepsilon \to \mathrm{D}(F_\varepsilon(x), F_0(x))$, $x \in X$, *are equi-continuous at* 0 *and equi-locally Lipschitz on* $(0, +\infty)$. *If F is K-ball-Lipschitz l.s.c. for some* $K \geqslant 1$ *then these mappings are equi-Lipschitz with constant K on* $[0, +\infty)$.

*Remark.* Proposition 3.3 specified to the case of arbitrary set $X$ (equipped with the trivial topology $\{\emptyset, X\}$ gives the following property of the intersection of convex sets.

3.6. Corollary. *Let* $\{F(x) \mid x \in X\}$ *be a family of closed convex sets in Y and assume that* $\bigcap_{x \in X} F(x) \neq \emptyset$. *Then the mapping* $\varepsilon \to \bigcap_{x \in X} F(x) - \varepsilon B^\circ$ *of* $[0, +\infty)$ *into Y has a convex graph and is locally Lipschitz on* $(0, +\infty)$, *with respect to the Hausdorff distance in* $\mathcal{N}(Y)$.

## 4. Approximation of Continuous Selections

Invoking Michael's continuous selection theorem, in this section we give simultaneously some extensions of this theorem and some information about the approximation of the set of continuous selections for a set valued map by the sets of $\varepsilon$-approximation selections.

In the space $C(X, Y)$ of continuous mappings of $X$ into $Y$ we consider the generalized sup-norm $\|f\|_\infty = \sup_x \|f(x)\|$ ($\|f\|_\infty = +\infty$ whenever $f$ is an unbounded mapping). We denote by $d(f, K)$ the distance from $f$ to a subset $K$ of $C(X, Y)$ and by $\mathcal{D}(L, K)$ the Hausdorff distance between two subsets $L$, $K$ of $C(X, Y)$.

We begin with a simple observation.

4.1. Proposition. *Assume that* $0 \leqslant \delta < \varepsilon$. *If a set valued mapping* $F: X \to \mathcal{N}(Y)$ *has closed values then*

$$C_{F_\delta} \subseteq C_F^\delta \subseteq C_{F_\varepsilon} \qquad \textit{and} \qquad C_{F_0} = C_F = C_F^0 = \bigcap_{\varepsilon > 0} C_F^\varepsilon.$$

*Proof.* We have $C_{F_\delta} \subseteq C_F^\delta$, since $F_\delta(x) \subseteq F(x) - \delta B^\circ$ for all $x \in X$. Also, $C_F^\delta \subseteq C_{F_\varepsilon}$, since for $f \in C_F^\delta$ and for all $x \in X$ it holds that $f(x) \in f_{\varepsilon - \delta}(x) \subseteq F_\varepsilon(x)$. The second assertion follows directly from the first one. Q.E.D.

In the subsequent lemmas and corollaries we assume that $X$ is a paracompact Hausdorff topological space, $Y$ is a Banach space, and

$F: X \to \mathcal{N}(Y)$ is an almost l.s.c. set valued mapping with closed convex values.

4.2. Lemma. *$C_{F_\varepsilon}(x) = F_\varepsilon(x)$ for all $x \in X$, $\varepsilon > 0$. If $F_0(x) \neq \emptyset$ for all $x \in X$ and $F_0$ is l.s.c. then also $C_{F_0}(x) = F_0(x)$ for all $x \in X$.*

*Proof.* Obviously $C_{F_\varepsilon}(x) \subseteq F_\varepsilon(x)$. For the reverse inclusion, recall that $F_\varepsilon$ has open lower sections (Proposition 2.1(4)). Therefore by the well-known Michael construction $F_\varepsilon$ admits a continuous selection (see [18, Theorem 3.1]). Moreover, it is easy to see that a continuous selection can be constructed through each point of the graph of $F_\varepsilon$. By Michael's theorem, the same is true for $F_0$ if $F_0$ is l.s.c., since $F_0$ has nonempty closed convex values (Proposition 2.2, 2.3). Q.E.D.

4.3. Lemma. *$\mathscr{d}(f, C_{F_\varepsilon}) = \sup_x \mathrm{d}(f(x), C_{F_\varepsilon}(x))$ for every $\varepsilon \geqslant 0$ and every continuous mapping $f$.*

*Proof.* The assertion follows trivially whenever $C_{F_\varepsilon} = \emptyset$, so assume $C_{F_\varepsilon} \neq \emptyset$. Put $h = \sup_x \mathrm{d}(f(x), C_{F_\varepsilon}(x))$. Clearly $h \leqslant \mathscr{d}(f, C_{F_\varepsilon})$. On the other hand for every $\alpha > 0$ and for every $x \in X$ the set $G(x) = C_{F_\varepsilon}(x) \cap (f(x) + (h + \alpha) B^\circ)$ is nonempty. For every $\varepsilon > 0$, since the sets $C_{F_\varepsilon}(x) = F_\varepsilon(x)$ are convex and $F_\varepsilon$ has open lower sections, it follows that the set valued mapping $x \to G(x)$ has convex values and open lower sections, and therefore has a continuous selection, say $g$. For $\varepsilon = 0$, the set valued mapping $x \to C_{F_0}(x)$ has closed convex values and is l.s.c. Then the set valued mapping $x \to \overline{G(x)}$ has by Michael's theorem a continuous selection, say also $g$. In both cases $g \in C_{F_\varepsilon}$ and $\mathscr{d}(f, C_{F_\varepsilon}) \leqslant \|f - g\|_\infty \leqslant h + \alpha$. Consequently, $\mathscr{d}(f, C_{F_\varepsilon}) = h$. Q.E.D.

4.4. Lemma. *For every continuous mapping $f: X \to Y$ and for $\delta, \varepsilon > 0$ the following equalities hold.*

(1) $\mathrm{d}(f(x), C_F^\varepsilon(x)) = \mathrm{d}(f(x), F_\varepsilon(x))$ *for all* $x \in X$,

(2) $\mathscr{d}(f, C_F^\varepsilon) = \sup_x \mathrm{d}(f(x), F_\varepsilon(x))$,

(3) $\mathrm{D}(C_F^\varepsilon(x), C_F^\delta(x)) = \mathrm{D}(F_\varepsilon(x), F_\delta(x))$ *for all* $x \in X$,

(4) $\mathscr{D}(C_F^\varepsilon, C_F^\delta) = \sup_x \mathrm{D}(F_\varepsilon(x), F_\delta(x))$.

*If $F_0(x) \neq \emptyset$ for all $x \in X$ and $F_0$ is l.s.c., then the equalities* (1)–(4) *hold for all $\delta, \varepsilon \geqslant 0$.*

*Proof.* (1) By Proposition 4.1 and Lemma 4.2 we have

$$\mathrm{d}(f(x), C_F^\varepsilon(x)) \leqslant \mathrm{d}(f(x), C_{F_\varepsilon}(x)) = \mathrm{d}(f(x), F_\varepsilon(x)),$$

for all $x \in X$ and for every $\varepsilon > 0$. On the other hand, for arbitrary $\delta > \varepsilon$, Proposition 3.4, Lemma 4.2, and Proposition 4.1 yield

$$\begin{aligned} \mathrm{d}(f(x), F_\varepsilon(x)) &\leqslant \mathrm{d}(f(x), F_\delta(x)) + |\partial_\varepsilon^+ \mathrm{d}(f(x), F_\varepsilon(x))|\,(\delta - \varepsilon) \\ &= \mathrm{d}(f(x), C_{F\delta}(x)) + |\partial_\varepsilon^+ \mathrm{d}(f(x), F_\varepsilon(x))|\,(\delta - \varepsilon) \\ &\leqslant \mathrm{d}(f(x), C_F^\varepsilon(x)) + |\partial_\varepsilon^+ \mathrm{d}(f(x), F_\varepsilon(x))|\,(\delta - \varepsilon). \end{aligned}$$

Since $|\partial_\varepsilon^+ \mathrm{d}(f(x), F_\varepsilon(x))| < +\infty$ whenever $\varepsilon > 0$, then taking $\delta \searrow \varepsilon$ we conclude that

$$\mathrm{d}(f(x), F_\varepsilon(x)) \leqslant \mathrm{d}(f(x), C_F^\varepsilon(x)).$$

For $\varepsilon = 0$, by Proposition 4.1 and Lemma 4.2 we have

$$\mathrm{d}(f(x), C_F^0(x)) = \mathrm{d}(f(x), C_{F_0}(x)) = \mathrm{d}(f(x), F_0(x))$$

for all $x \in X$.

(2) Obviously $\sup_x \mathrm{d}(f(x), C_F^\varepsilon(x)) \leqslant \mathscr{d}(f, C_F^\varepsilon)$. By Proposition 4.1 and Lemmas 4.3 and 4.2 we have

$$\mathscr{d}(f, C_F^\varepsilon) \leqslant \mathscr{d}(f, C_{F_\varepsilon}) = \sup_x \mathrm{d}(f(x), C_{F_\varepsilon}(x)) = \sup_x \mathrm{d}(f(x), F_\varepsilon(x)).$$

(3) By (1)

$$\begin{aligned} \mathrm{D}(C_F^\varepsilon(x), C_F^\delta(x)) &= \sup_y |\mathrm{d}(y, C_F^\varepsilon(x)) - \mathrm{d}(y, C_F^\delta(x))| \\ &= \sup_y |\mathrm{d}(y, F_\varepsilon(x)) - \mathrm{d}(y, F_\delta(x))| \\ &= \mathrm{D}(F_\varepsilon(x), F_\delta(x)). \end{aligned}$$

(4) Obviously $\sup_x \mathrm{D}(C_F^\varepsilon(x), C_F^\delta(x)) \leqslant \mathscr{D}(C_F^\varepsilon, C_F^\delta)$. On the other hand

$$\begin{aligned} \mathscr{D}(C_F^\varepsilon, C_F^\delta) &= \sup_f |\mathscr{d}(f, C_F^\varepsilon) - \mathscr{d}(f, C_F^\delta)| \\ &\leqslant \sup_f \sup_x |\mathrm{d}(f(x), F_\varepsilon(x)) - \mathrm{d}(f(x), F_\delta(x))| \\ &\leqslant \sup_x \mathrm{D}(F_\varepsilon(x), F_\delta(x)) = \sup_x \mathrm{D}(C_F^\varepsilon(x), C_F^\delta(x)) \end{aligned}$$

(here $\sup_f$ means that supremum is taken over all $f \in C(X, Y)$). Q.E.D.

4.5. COROLLARY. *If*

$$\sup_x \lim_{\gamma \searrow 0} \frac{\mathrm{D}(F_\gamma(x), F_0(x))}{\gamma} < +\infty,$$

*then* $\mathscr{D}(C_F^\varepsilon, C_{F_\varepsilon}) = 0$ *for* $\varepsilon > 0$.

Indeed, by Proposition 4.1, Lemma 4.4(4), and Proposition 3.4, we have

$$\mathscr{D}(C_F^\varepsilon, C_{F_\varepsilon}) \leqslant \mathscr{D}(C_F^\varepsilon, C_F^\delta) = \sup_x \mathrm{D}(F_\varepsilon(x), F_\delta(x))$$
$$\leqslant \sup_x \lim_{\gamma \searrow 0} \frac{\mathrm{D}(F_\gamma(x), F_0(x))}{\gamma} |\varepsilon - \delta|,$$

whenever $0 \leqslant \delta < \varepsilon$. Letting $\delta \nearrow \varepsilon$ we obtain $\mathscr{D}(C_F^\varepsilon, C_{F_\varepsilon}) = 0$. Note that we have already established in Proposition 4.1 that $C_F^0 = C_F = C_{F_0}$.

Now we are ready to formulate the main results of this section.

4.6. THEOREM. *Let X be a paracompact Hausdorff topological space, Y be a Banach space. Assume that* $F: X \to \mathscr{N}(Y)$ *is a set valued mapping with closed convex values such that* $F(x) \neq Y$ *for some* $x \in X$, $F_0(x) \neq \varnothing$ *for all* $x \in X$ *and* $F_0$ *is l.s.c. Then* $C_F \neq \varnothing$ *and*

$$\varepsilon \leqslant \mathscr{D}(C_F^\varepsilon, C_F) \leqslant \sup_x \lim_{\gamma \searrow 0} \frac{\mathrm{D}(F_\gamma(x), F_0(x))}{\gamma} \varepsilon$$

*for* $\varepsilon \geqslant 0$.

*Proof.* Since $F_0$ has closed convex values, $C_{F_0} \neq \varnothing$ by Michael's theorem, and $C_F = C_{F_0}$ by Proposition 4.1. The inequalities follow directly from Lemma 4.4(4) and Proposition 3.4. Q.E.D.

By Theorem 3.1, Theorem 4.6, Lemma 4.4, and Corollary 3.5, we have the following corollary.

4.7. COROLLARY. *Assume that* $F: X \to \mathscr{N}(Y)$ *is a set valued mapping with closed convex values and* $F(x) \neq Y$ *for some* $x \in X$. *If F is ball-locally-uniformly l.s.c. then* $C_F \neq \varnothing$ *and for every continuous mapping* $f: X \to Y$ *and every* $x \in X$ *it holds that* $\mathrm{d}(f(x'), C_F^\varepsilon(x')) \to \mathrm{d}(f(x'), C_F(x'))$ *as* $\varepsilon \to 0$, *uniformly for all x' from some neighborhood U of x. Moreover, if F is ball-uniformly l.s.c. then* $\varepsilon \leqslant \mathscr{D}(C_F^\varepsilon, C_F) \to 0$ *as* $\varepsilon \to 0$ *and the function* $\varepsilon \to \mathscr{D}(C_F^\varepsilon, C_F)$ *is locally Lipschitz on* $(0, +\infty)$. *If F is K-ball-Lipschitz l.s.c. for some* $K \geqslant 1$, *then* $\varepsilon \leqslant \mathscr{D}(C_F^\varepsilon, C_F) \leqslant K\varepsilon$ *for* $\varepsilon \geqslant 0$.

## 5. Localization of Continuous Selections

In this section we discuss the lower semicontinuity concepts for the intersection of a given set valued mapping with a ball valued multifunction. In particular, we answer the question: In what cases is the intersection of a ball-Lipschitz l.s.c. set valued mapping with a ball valued multifunction ball-Lipschitz l.s.c.? As an effect one can obtain more detailed information on localization of continuous selections for set valued mappings.

For a pair of mappings $f: X \to Y$, $d: X \to [0, +\infty)$, and $L \geqslant 0$, we denote by $B^L$ the subordinated ball valued multifunction defined by $B^L(x) = f(x) + Ld(x)B$. We begin with a simple general result.

5.1. Proposition. *Assume that $f: X \to Y$ and $d: X \to [0, +\infty)$ are continuous mappings and $F: X \to \mathcal{N}(Y)$ is a set valued mapping with closed convex values such that $F_0(x) \neq \emptyset$ for all $x \in X$ and $F_0$ is l.s.c. If $F_0(x) \cap (f(x) + d(x)B) \neq \emptyset$ for all $x \in X$, then for every $L > 1$ the set valued mapping $F \cap B^L$ defined by the formula*

$$F \cap B^L(x) = F(x) \cap (f(x) + Ld(x)B)$$

*is such that $(F \cap B^L)_0$ is l.s.c. and $(F \cap B^L)_0 = F_0 \cap B^L$.*

*Proof.* Since $f$ and $d$ are continuous, $B^L$ is l.s.c. It is known that lower semicontinuity of $F_0$ and $B^L$, together with the nonemptiness of $F_0(x) \cap B^1(x)$ for all $x \in X$, implies that $F_0 \cap B^L$ is l.s.c. (see [13, Lemma 7.1; 14, Proposition 2]). Therefore by Propositions 2.1(8) and 2.11,

$$(F \cap B^L)_0 \subseteq F_0 \cap (B^L)_0 = F_0 \cap B^L = (F_0 \cap B^L)_0 \subseteq (F \cap B^L)_0. \quad \text{Q.E.D.}$$

Now we pass to ball-Lipschitz l.s.c. set valued mappings.

5.2. Lemma. *Assume that $f: X \to Y$ and $d: X \to [0, +\infty)$ are continuous mappings and $K \geqslant 1$. If $F: X \to \mathcal{N}(Y)$ is K-ball-Lipschitz l.s.c. and $F(x) \cap B^1(x) \neq \emptyset$ for all $x \in X$, then $F_\mu(x) \cap B^K(x) \neq \emptyset$ for all $x \in X$ and $\mu > 0$.*

*Proof.* By the continuity of $f$ and $d$, every $x \in X$ has a neighborhood $U$ such that

$$B^1(x') \subseteq B^1(x) + (3K)^{-1} \mu B$$

for all $x' \in U$. Since $F(x') \cap B^1(x') \neq \emptyset$ for all $x' \in X$, then $0 \in F_{B^1(x) + (3K)^{-1}\mu B}(x)$. Hence, by the definition of K-ball-Lipschitz l.s.c.,

$$F_{2^{-1}\mu}(x) \cap (B^K(x) + 3^{-1}\mu B) \neq \emptyset,$$

or equivalently,

$$(F_{2^{-1}\mu}(x) - 3^{-1}\mu B) \cap B^K(x) \neq \varnothing.$$

Since $F_{2^{-1}\mu}(x) - 3^{-1}\mu B \subseteq F_\mu(x)$ (Proposition 2.1), we get the assertion.

Q.E.D.

The next lemma is a refinement of Corollary 1 in [15].

5.3. LEMMA. *Assume that $E$ is a convex subset of a normed linear space $Y$, $y \in Y$ and $r \geqslant 0$. If $E \cap (y + rB^\circ) \neq \varnothing$, then*

$$(E + \varepsilon B^\circ) \cap (y + LrB + \varepsilon B^\circ) \subset E \cap (y + LrB) + N(L)\, \varepsilon B^\circ,$$

*for every $\varepsilon > 0$, $L > 1$, where $N(L) = 1 + 2(L+1)/(L-1)$.*

*Proof.* If $r = 0$ then the inclusion holds trivially, so we assume that $r > 0$. Without loss of generality we may also assume that $y = 0$. Thus we should prove that

$$(E + \varepsilon B^\circ) \cap (LrB + \varepsilon B^\circ) \subset E \cap (LrB) + N(L)\, \varepsilon B^\circ,$$

whenever $r > 0$, $\varepsilon > 0$, $L > 1$. Let

$$z \in (E + \varepsilon B^\circ) \cap (LrB + \varepsilon B^\circ) = (E + \varepsilon B^\circ) \cap (Lr + \varepsilon)\, \mathrm{B}^\circ.$$

Then there exists an element $v \in E$ such that $\|v - z\| < \varepsilon$. Clearly $\|v\| \leqslant \|z\| + \varepsilon < Lr + 2\varepsilon$. If $\|v\| \leqslant Lr$, then obviously $z \in E \cap (LrB) + N(L)\, \varepsilon B^\circ$, so the only nontrivial case is $\|v\| > Lr$. By the assumption, there exists an element $v' \in E$ such that $\|v'\| \leqslant r < Lr$. Choose $\lambda \in (0, 1)$ such that for $v_\lambda = \lambda v' + (1 - \lambda) v$ it holds that $\|v_\lambda\| = Lr$. By the convexity of $E$ we have $v_\lambda \in E \cap (LrB)$. By the triangle inequality,

$$Lr \leqslant \lambda r + (1 - \lambda)(Lr + 2\varepsilon).$$

Hence $\lambda \leqslant 2\varepsilon((L-1)r + 2\varepsilon)^{-1}$. Consequently,

$$\begin{aligned} \|v - v_\lambda\| &= \lambda \|v - v'\| \leqslant \lambda(\|v\| + \|v'\|) \\ &\leqslant 2\varepsilon((L-1)r + 2\varepsilon)^{-1}\, ((L+1)r + 2\varepsilon) \\ &\leqslant 2\varepsilon(L+1)/(L-1). \end{aligned}$$

Finally,

$$\|z - v_\lambda\| \leqslant \|z - v\| + \|v - v_\lambda\| < \varepsilon + 2\varepsilon(L+1)/(L-1).$$

Thus $z \in E \cap (LrB) + N(L)\, \varepsilon B^\circ$. Q.E.D.

5.4. THEOREM. *Assume that* $f\colon X\to Y$ *and* $d\colon X\to[0,+\infty)$ *are continuous mappings and* $K\geqslant 1$. *If* $F\colon X\to\mathcal{N}(Y)$ *is* $K$*-ball-Lipschitz l.s.c., has convex values and* $F(x)\cap(f(x)+d(x)B)\neq\varnothing$ *for all* $x\in X$, *then for every* $L>1$ *the set valued mapping* $F\cap B^{LK}$ *defined by the formula* $F\cap B^{LK}(x)=F(x)\cap(f(x)+LKd(x)B)$ *is* $K'$*-ball-Lipschitz l.s.c. for some* $K'\leqslant(1+2(L+1)/(L-1))K$.

*Proof.* Let us take an arbitrary $\mu>0$. Since $F(x)\cap B^1(x)\neq\varnothing$ for all $x\in X$ and $LK>1$, then by Lemma 5.3

$$(F(x)-\mu B^\circ)\cap(B^{LK}(x)-\mu B^\circ)\subseteq(F(x)\cap B^{LK}(x))+N(LK)\,\mu B^\circ$$

for all $x\in X$. From this it follows easily that

$$F_\mu(x)\cap(B^{LK})_\mu(x)\subseteq(F\cap B^{LK})_{N(LK)\mu}(x).$$

But $B^{LK}$ is l.s.c., since $f$ and $d$ are continuous. Thus, by Proposition 2.11, for all $x\in X$

$$(B^{LK})_\mu(x)=B^{LK}(x)-\mu B^\circ.$$

Hence, for all $x\in X$ we have

$$F_\mu(x)\cap(B^{LK}(x)-\mu B^\circ)\subseteq(F\cap B^{LK})_{N(LK)\mu}(x).$$

Taking into account that by Lemma 5.2

$$\varnothing\neq F_\mu(x)\cap B^K(x)\subseteq F_\mu(x)\cap B^{LK}(x),$$

we get $(F\cap B^{LK})_{N(LK)\mu}(x)\neq\varnothing$. Since $\mu$ was arbitrary we conclude that $(F\cap B^{LK})_\varepsilon(x)\neq\varnothing$ for all $x\in X$ and for every $\varepsilon>0$. On the other hand, by Proposition 2.1(8), it is easy to see that

$$(F\cap B^{LK})_\varepsilon\subseteq F_\varepsilon\cap(B^{LK})_\varepsilon.$$

Therefore, since $B^{LK}$ is l.s.c. and $F$ is $K$-ball-Lipschitz l.s.c., we have

$$\begin{aligned}\varnothing\neq(F\cap B^{LK})_\varepsilon(x)&\subseteq F_\varepsilon(x)\cap(B^{LK})_\varepsilon(x)=F_\varepsilon(x)\cap(B^{LK}(x)-\varepsilon B^\circ)\\&\subseteq(F_\mu(x)-K\varepsilon B^\circ)\cap(B^{LK}(x)-K\varepsilon B^\circ).\end{aligned}$$

Again by Lemma 5.3,

$$(F_\mu(x)-K\varepsilon B^\circ)\cap(B^{LK}(x)-K\varepsilon B^\circ)\subseteq(F_\mu(x)\cap B^{LK}(x))+N(L)\,K\varepsilon B^\circ.$$

But

$$F_\mu(x)\cap B^{LK}(x)\subseteq F_\mu(x)\cap(B^{LK}(x)-\mu B^\circ)\subseteq(F\cap B^{LK})_{N(LK)\mu}(x).$$

Therefore

$$\emptyset \neq (F \cap B^{LK})_{\varepsilon}(x) \subseteq (F \cap B^{LK})_{N(LK)\mu}(x) + N(L)\, K\varepsilon B^{\circ}$$

for all $x \in X$ and for every $\varepsilon > 0$. Since $\mu$ was chosen arbitrarily, this ensures that $F \cap B^{LK}$ is $K'$-ball-Lipschitz l.s.c. for $K' \leqslant N(L) K = (1 + 2(L+1)/(L-1))K$. Q.E.D.

*Remark.* If $Y$ is a Banach space, then under the assumptions of the above theorem the set valued mapping $(F \cap B^{LK})_0$ has nonempty values, is l.s.c. and $(F \cap B^{LK})_0 = F_0 \cap B^{LK}$.

For the concepts of weak lower semicontinuity and convex lower semicontinuity the following sharper conclusion is true (see [15]).

5.5. PROPOSITION. *Under the assumptions of Theorem* 5.4, *but with $F$ weak (respectively: convex) l.s.c., for every $L > 1$ the set valued mapping $F \cap B^L$ defined by the formula $F \cap B^L(x) = F(x) \cap (f(x) + Ld(x)B)$ is weak (respectively: convex) l.s.c.*

## 6. REMARK ON SELECTION EXTENSION PROPERTY

Let $X$ be a paracompact Hausdorff topological space, $Y$ be a Banach space, and $F: X \to \mathcal{N}(Y)$ be a set valued mapping with closed convex values. It was established by Michael [13] that $F$ is l.s.c. if and only if for every closed subset $A$ of $X$, every continuous selection $f: A \to Y$ of $F|_A$ extends to a continuous selection $\bar{f}: X \to Y$ of $F$. The direct consequence of this selection extension property is that $F$ is l.s.c. if and only if $F(x) = C_F(x)$ for all $x \in X$. It means that, in general, the selection extension property as well as the last equality no longer holds for set valued mappings satisfying continuity conditions considered in our paper. Clearly all these conditions guarantee that $F_0(x) = C_F(x)$ for all $x \in X$, since they imply the lower semicontinuity of $F_0$. Let us point out that they are also sufficient for the following weaker extension property.

6.1. PROPOSITION. *Let $X$ be a normal topological space and $Y$ be a topological vector space. If a set valued mapping $F: X \to \mathcal{N}(Y)$ has convex values and admits a continuous selection, then for every open set $W \subseteq X$, every closed set $D \subseteq W$, and any continuous selection $f: W \to Y$ of $F|_W$ there exists a continuous selection $\bar{f}: X \to Y$ of $F$ such that $\bar{f}|_D = f|_D$.*

*Proof.* Since $D$ and $X \setminus W$ are disjoint closed sets, there exist two open sets $U$, $V$, such that

$$D \subset U, \qquad X \setminus W \subset V, \qquad \bar{U} \cap \bar{V} = \emptyset.$$

By the Urysohn lemma there exists a continuous mapping $\alpha: X \to [0, 1]$ such that $\alpha(x) = 1$ for $x \in \bar{U}$ and $\alpha(x) = 0$ for $x \in \bar{V}$. Let $g$ be any continuous selection of $F$. The selection $\tilde{f}$ may be defined by the formula

$$\tilde{f}(x) = \alpha(x) f(x) + (1 - \alpha(x)) g(x). \qquad \text{Q.E.D.}$$

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